

Particle-Like Solutions of the Einstein-Dirac-Maxwell Equations

Felix Finster*, Joel Smoller† and Shing-Tung Yau‡

February 1998

Abstract

We consider the coupled Einstein-Dirac-Maxwell equations for a static, spherically symmetric system of two fermions in a singlet spinor state. Soliton-like solutions are constructed numerically. The stability and the properties of the ground state solutions are discussed for different values of the electromagnetic coupling constant. We find solutions even when the electromagnetic coupling is so strong that the total interaction is repulsive in the Newtonian limit. Our solutions are regular and well-behaved; this shows that the combined electromagnetic and gravitational self-interaction of the Dirac particles is finite.

1 Introduction

In the recent paper [1], we studied the coupled Einstein-Dirac (ED) equations for a spherically symmetric system of two fermions in a singlet spinor state. Using numerical methods, we found particle-like solutions and analyzed their properties. In this letter, we extend our results to the more physically relevant system where the Einstein-Dirac equations are coupled to an electromagnetic field. We find particle-like solutions which are linearly stable (with respect to spherically symmetric, time-dependent perturbations), and have other interesting properties.

The general Einstein-Dirac-Maxwell (EDM) equations for a system of n Dirac particles are

$$R_j^i - \frac{1}{2} R \delta_j^i = -8\pi T_j^i \quad , \quad (G - m) \Psi_a = 0 \quad , \quad \nabla_k F^{jk} = 4\pi e \sum_{a=1}^n \overline{\Psi}_a G^j \Psi_a \quad , \quad (1.1)$$

where T_j^i is the sum of the energy-momentum tensor of the Dirac particles and the Maxwell stress-energy tensor. Here the G^j are the Dirac matrices, which are related to the Lorentzian metric via the anti-commutation relations

$$g^{jk}(x) = \frac{1}{2} \{ G^j(x), G^k(x) \} \quad ,$$

F_{jk} is the electromagnetic field tensor, and Ψ_a are the wave functions of fermions of mass m and charge e . The Dirac operator is denoted by G ; it depends on both the gravitational and

*Research supported by the Deutsche Forschungsgemeinschaft and the Schweizerischer Nationalfonds.

†Research supported in part by the NSF, Grant No. DMS-G-9501128.

‡Research supported in part by the NSF, Grant No. 33-585-7510-2-30.

the electromagnetic field (for details, see e.g. [2]). In order to get a spherically symmetric system, we consider, as in [1], two fermions with opposite spin, i.e. a singlet spinor state. Using the ansatz in [1], we reduce the Dirac 4-spinors to a 2-component real spinor system (α, β) . We show numerically that the EDM equations also have particle-like solutions, which are characterized by the “rotation number” $n = 0, 1, \dots$ of the vector (α, β) . In contrast to [1], we restrict our attention to studying the case $n = 0$, the ground state; this case illustrates quite nicely the physical effects due to the addition of the electromagnetic interaction. We anticipate that the situation for solutions with higher rotation number will be qualitatively similar.

The relative coupling of the electromagnetic and the gravitational field is described by the parameter $(e/m)^2$. If we consider the ground state solutions for fixed value of $(e/m)^2$, we find that the mass-energy spectrum (i.e., the plot of the binding energy of the fermions vs. the rest mass) is a spiral curve which tends to a limiting configuration Γ . This implies the interesting result that for parameter values on Γ , there exist an infinite number of ($n = 0$)-solutions (the one of lowest energy is the ground state), while for parameter values near Γ , there are a large, but finite number of such solutions. For small coupling (i.e., for small m and $(e/m)^2 < 1$), our solutions are linearly stable with respect to spherically symmetric perturbations. If we compare the (ADM) mass ρ with the total rest mass $2m$, we find that for small m , the total binding energy $\rho - 2m$ is negative, implying that energy is gained in forming the singlet state. It is thus physically reasonable that such states should be stable. However, the stable solutions become unstable as the binding energy of the fermions increases; this is shown using Conley index methods together with bifurcation theory (see [3, Part IV]).

In order to study the effect of the electromagnetic interaction in more detail, we look at the behavior of the solutions as the parameter $(e/m)^2$ is varied. For weak electromagnetic coupling $(e/m)^2 \ll 1$, the solutions are well-behaved and look similar to the solutions of the ED equations [1]. For $(e/m)^2 > 1$, the form of the solutions changes drastically. In a simplified argument, this can already be understood from the nonrelativistic, classical limit of the EDM equations. Namely, according to Newton’s and Coulomb’s laws, the force between two charged, massive point particles has the well-known form

$$F = -\frac{m^2}{r^2} + \frac{e^2}{r^2} \quad (1.2)$$

(we work in standard units $\hbar = c = \mathcal{G} = 1$). For $(e/m)^2 < 1$, the gravitational attraction dominates the electromagnetic repulsion, and it therefore seems reasonable that we get bound states. For $(e/m)^2 > 1$, however, the total force is repulsive, and classically one can no longer expect bound states. The EDM equations, however, do have solutions even for $(e/m)^2 > 1$. This is a surprising effect which can again only be explained by the nonlinearity of Einstein’s equations. For such solutions to exist, however, the rest mass m of the fermions must be sufficiently large.

We remark that there are related works [5, 6, 7], where the authors obtain soliton solutions for the Dirac-Maxwell equations (in the absence of gravity), but these solutions, unfortunately, have the undesirable feature of having negative energy.

2 The Equations

We choose polar coordinates $(t, r, \vartheta, \varphi)$ and write the metric in the form

$$ds^2 = \frac{1}{T^2} dt^2 - \frac{1}{A} dr^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2$$

with positive functions $A(r), T(r)$. Using the ansatz from [1, Eqns. (3.4),(3.6)], we describe the Dirac spinors with two real functions α, β . For the derivation of the corresponding EDM equations, we simply modify the ED equations [1, Eqns. (5.4)-(5.8)]. In the Dirac equation, the electromagnetic field is introduced by the minimal coupling procedure $\partial_j \rightarrow \partial_j - ie\mathcal{A}_j$, where e is the unit charge and \mathcal{A} the electromagnetic potential (see [4]). In the static case, the fermions only generate an electric field; thus we can assume that the electromagnetic potential has the form $\mathcal{A} = (-\phi, \vec{0})$ with the Coulomb potential ϕ . Since the time-dependence of the wave functions is a plane wave $\exp(-i\omega t)$, minimal coupling reduces to the replacement $\omega \rightarrow \omega - e\phi$. Thus the Dirac equations are

$$\sqrt{A} \alpha' = \frac{1}{r} \alpha - ((\omega - e\phi)T + m) \beta \quad (2.1)$$

$$\sqrt{A} \beta' = ((\omega - e\phi)T - m) \alpha - \frac{1}{r} \beta \quad (2.2)$$

In the Einstein equations, we must include the stress-energy tensor of the electric field [8]; this gives

$$r A' = 1 - A - 16\pi(\omega - e\phi)T^2 (\alpha^2 + \beta^2) - r^2 AT^2 (\phi')^2 \quad (2.3)$$

$$2rA \frac{T'}{T} = A - 1 - 16\pi(\omega - e\phi)T^2 (\alpha^2 + \beta^2) + 32\pi \frac{1}{r} T \alpha\beta + 16\pi mT (\alpha^2 - \beta^2) + r^2 AT^2 (\phi')^2 \quad (2.4)$$

Using current conservation, Maxwell's equations $\nabla_k F^{jk} = 4\pi e \sum_{a=1}^2 \overline{\Psi}_a G^j \Psi_a$ reduce to the single second-order differential equation

$$r^2 A \phi'' = -8\pi e (\alpha^2 + \beta^2) - \left(2rA + r^2 A \frac{T'}{T} + \frac{r^2}{2} A' \right) \phi' \quad (2.5)$$

The normalization condition for the wave functions [1, Eqn. (5.8)] remains unchanged, namely

$$\int_0^\infty (\alpha^2 + \beta^2) \frac{T}{\sqrt{A}} dr = \frac{1}{4\pi} \quad (2.6)$$

We seek smooth solutions of these equations, which are asymptotically Minkowskian and have finite (ADM) mass ρ ,

$$\lim_{r \rightarrow \infty} T(r) = 1 \quad (2.7)$$

$$\rho := \lim_{r \rightarrow \infty} \frac{r}{2} (1 - A(r)) < \infty \quad (2.8)$$

Furthermore, we demand that the electromagnetic potential vanishes at infinity,

$$\lim_{r \rightarrow \infty} \phi(r) = 0 \quad (2.9)$$

The equations (2.1)-(2.5) are invariant under the gauge transformations

$$\phi(r) \rightarrow \phi(r) + \kappa, \quad \omega \rightarrow \omega + e\kappa, \quad \kappa \in \mathbb{R}. \quad (2.10)$$

3 Construction of the Solutions

The construction of solutions is, (analogous to [1]), simplified by the following scaling argument. We weaken the conditions (2.6), (2.7), (2.9) to

$$0 \neq \int_0^\infty (\alpha^2 + \beta^2) \frac{T}{\sqrt{A}} dr < \infty, \quad 0 \neq \lim_{r \rightarrow \infty} T(r) < \infty, \quad \lim_{r \rightarrow \infty} \phi(r) < \infty, \quad (3.1)$$

and set instead

$$T(0) = 1, \quad \phi(0) = 0, \quad m = 1. \quad (3.2)$$

This simplifies the discussion of the equations near $r = 0$; indeed, a Taylor expansion around the origin gives

$$\begin{aligned} \alpha(r) &= \alpha_1 r + \mathcal{O}(r^2), & \beta(r) &= \mathcal{O}(r^2) \\ A(r) &= 1 + \mathcal{O}(r^2), & T(r) &= 1 + \mathcal{O}(r^2) \\ \phi(r) &= \mathcal{O}(r^2). \end{aligned}$$

The solutions are now determined by only three real parameters e , ω , and α_1 . For a given value of these parameters, we can construct initial data at $r = 0$ and, using the standard Mathematica ODE solver, we shoot for numerical solutions of the modified system (2.1)-(2.5), (3.2). By varying ω (for fixed e and α_1), we can arrange that the spinors (α, β) tend to the origin for large r . As one sees numerically, the so-obtained solutions satisfy the conditions (2.8) and (3.1).

For a given solution $(\alpha, \beta, A, T, \phi)$ of this modified system, we introduce the scaled functions

$$\begin{aligned} \tilde{\alpha}(r) &= \sqrt{\frac{\tau}{\lambda}} \alpha(\lambda r), & \tilde{\beta}(r) &= \sqrt{\frac{\tau}{\lambda}} \beta(\lambda r) \\ \tilde{A}(r) &= A(\lambda r), & \tilde{T}(r) &= \tau^{-1} T(\lambda r) \\ \tilde{\phi}(r) &= \tau \phi(\lambda r). \end{aligned}$$

As one verifies by direct computation, these functions satisfy the original equations (2.1)-(2.8) if the physical parameters are transformed according to

$$\tilde{m} = \lambda m, \quad \tilde{\omega} = \lambda \tau \omega, \quad \tilde{e} = \lambda e,$$

where the scaling factors λ and τ are given by

$$\lambda = \left(4\pi \int_0^\infty (\alpha^2 + \beta^2) \frac{T}{\sqrt{A}} dr \right)^{\frac{1}{2}}, \quad \tau = \lim_{r \rightarrow \infty} T(r).$$

Then the condition (2.9) can be fulfilled by a suitable gauge transformation (2.10). Notice that $(\tilde{e}/\tilde{m})^2 = e^2$ is invariant under the scaling. Therefore it is convenient to take $(\tilde{e}/\tilde{m})^2$ (and not \tilde{e}^2 itself) as the parameter to describe the strength of the electromagnetic coupling.

We point out that this scaling technique is merely used to simplify the numerics; for the physical interpretation, however, one must always work with the scaled tilde solutions. Since the transformation from the un-tilde to the tilde variables is one-to-one, our scaling method yields all the solutions of the original system. We will from now on consider only the *scaled* solutions; for simplicity in notation, the tilde will be omitted.

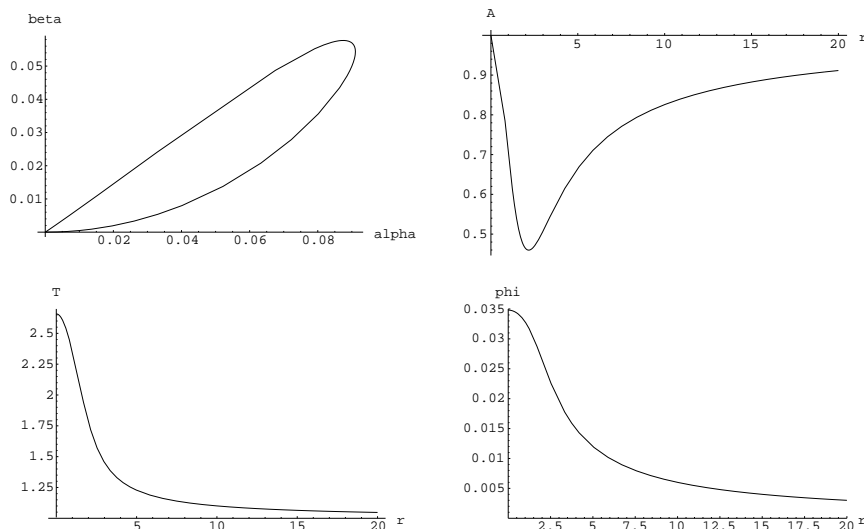


Figure 1: Solution of the EDM equations for parameter values $(e/m)^2 = 0.7162$, $m = 0.7639$, $\omega = 0.6807$, $\rho = 1.15416$ ($\alpha'(0) = 0.05361$).

4 Properties of the Solutions

The solutions we found have different rotation number $n = 0, 1, \dots$ of the vector (α, β) . In the nonrelativistic limit, n coincides with the number of zeros of the corresponding Schrödinger wave functions, and thus $n = 0$ corresponds to the ground state, $n = 1$ to the first excited state, \dots , etc. Because of the nonlinearity of our equations, n does not in general have this simple interpretation. In the following, we will restrict ourselves to the $n = 0$ solutions. A plot of a typical solution is shown in Figure 1.

For all considered solutions, the spinors (α, β) decay exponentially at infinity. This means physically that the fermions have a high probability of being confined to a neighborhood of the origin. Since the spinors decay so rapidly at infinity, our solutions asymptotically go over into spherically symmetric solutions of the Einstein-Maxwell equations; i.e. the Reissner-Nordström solution [8]. More precisely, the behavior for large r is

$$A(r) \approx T(r)^{-2} \approx 1 - \frac{2\rho}{r} + \frac{(2e)^2}{r^2} \quad , \quad \phi(r) \approx \frac{2e}{r} \quad .$$

Thus asymptotically, our solution looks like the gravitational and electrostatic field generated by a point particle at the origin having mass ρ and charge $2e$. In contrast to the Reissner-Nordström solution, however, our solutions have no event horizons or singularities. This can be understood from the fact that we consider quantum mechanical particles (instead of point particles), which implies that the wave functions are delocalized according to the Heisenberg Uncertainty Principle. As a consequence, the distribution of matter and charge are also delocalized, and this prevents the metric from forming singularities.

The situation near the origin $r = 0$, on the other hand, is parametrized by the rest mass m and the energy ω of the fermions. In Figure 2, the binding energy $m - \omega$ is plotted versus m for different values of the parameter $(e/m)^2$. One sees that $m - \omega$ is always positive, which means that the fermions are in a bound state. For weak electromagnetic coupling (see Figure 2, plots A and B), the curve has the form of a spiral which starts at

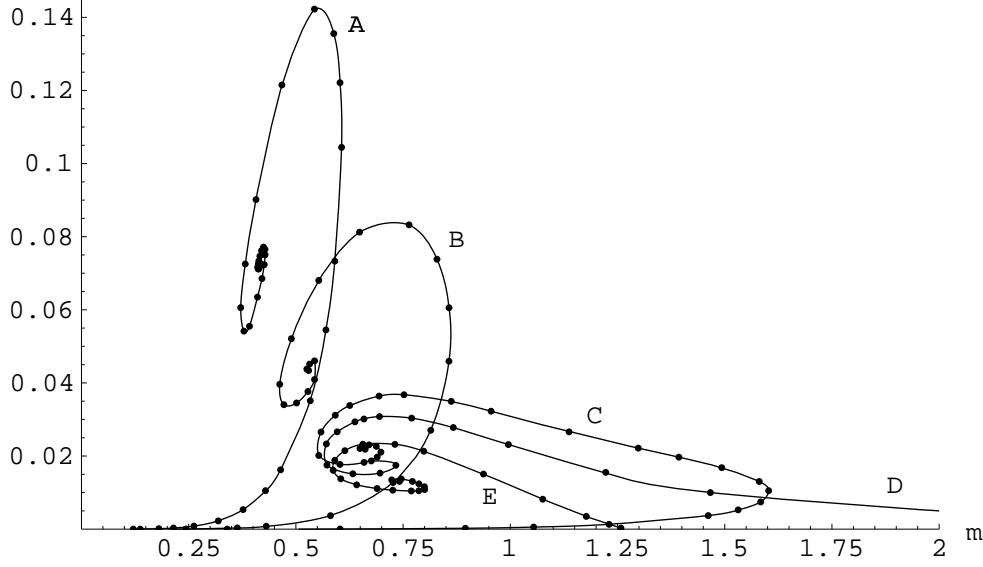


Figure 2: Binding Energy $m - \omega$ of the Fermions for $(e/m)^2 = 0$ (A), 0.7162 (B), 0.9748 (C), 1 (D), and 1.0313 (E).

the origin. The binding energy becomes smaller for fixed m and increasing $(e/m)^2$; this is because the electromagnetic repulsion weakens the binding. The mass-energy spectrum when $(e/m)^2 \ll 1$ has a similar shape as in the case without the electromagnetic interaction [1]. The stability techniques and results of [1] can be generalized directly: For small m , one can use linear perturbation theory to show numerically that the solutions are stable (with respect to spherically symmetric perturbations). The stability for larger values of m can be analyzed with Conley index theory (see [3]), where we take m as the bifurcation parameter. The Conley index theory yields that the stability/instability of a solution remains unchanged if the parameter m is continuously varied and no bifurcations take place. Moreover, at the bifurcation points, the Conley index allows us to analyze the change of stability with powerful topological methods. We find that all the solutions on the “lower branch” of the spiral (i.e., on the curve from the origin up to the maximal value of m) are stable, whereas all the solutions on the “upper branch” are unstable.

The form of the energy spectrum changes when $(e/m)^2 \approx 1$. This is the regime where the electrostatic and gravitational forces balance each other in the classical limit (1.2). Since the assumption of classical point particles does not seem to be appropriate for our system, we replace (1.2) by taking the nonrelativistic limit of the EDM equations more carefully. For this, we assume that m and e are small (for fixed $(e/m)^2$). The coupling becomes weak in this limit; thus $A, T \approx 1$, $\phi \approx 0$. The Dirac equations give $\omega \approx m$ and $\alpha \gg \beta$. Therefore the EDM equations go over into the Schrödinger equation with the Newtonian and Coulomb potentials,

$$\begin{aligned} \left(-\frac{1}{2m} \Delta + e\phi + mV \right) \Psi &= E \Psi \\ -\Delta V &= -8\pi m |\Psi|^2, \quad -\Delta \phi = 8\pi e |\Psi|^2 \end{aligned}$$

(with $E = \omega - m$, $\Psi(r) = \alpha(r)/r$, $V = 1 - T$; Δ equals the radial Laplacian in \mathbb{R}^3). One sees from these equations that, as in (1.2), the Newtonian and Coulomb potentials are just multiples of each other; namely $V = -m\phi/e$. For $(e/m)^2 > 1$, the total interaction

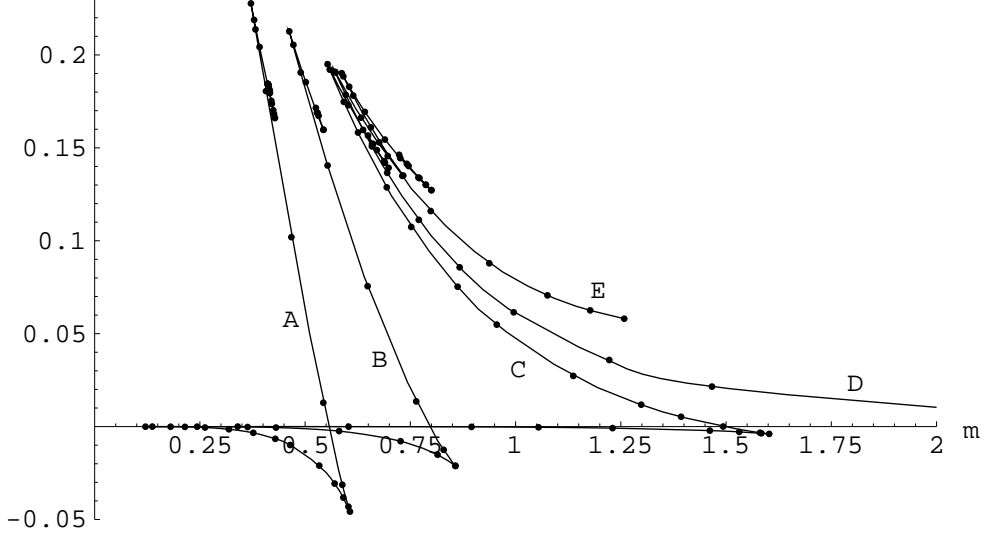


Figure 3: Total Binding Energy $\rho - 2m$ for $(e/m)^2 = 0$ (A), 0.7162 (B), 0.9748 (C), 1 (D), and 1.0313 (E).

is repulsive, and the Schrödinger equation has no bound states. We conclude that, in the limit of small m , the EDM equations cannot have particle-like solutions if $(e/m)^2 > 1$. In other words, the mass-energy plots of Figure 2 can only start at $m = 0$ if $(e/m)^2 < 1$. This is confirmed by the numerics (see Figure 2, plots C, D, and E). For $(e/m)^2 = 1$, the plot tends asymptotically to the point $(m = \infty, m - \omega = 0)$. It is surprising that we still have bound states for $(e/m)^2 > 1$. In this case however, there are only solutions if m is sufficiently large and smaller than some threshold value where the binding energy of the fermions becomes zero.

In Figure 3, the total binding energy $\rho - 2m$ is plotted for different values of $(e/m)^2$. For $(e/m)^2 < 1$, $\rho - 2m$ is negative for the stable solutions. For $(e/m)^2 > 1$, however, $\rho - 2m$ is always positive. This indicates that these solutions should be unstable, since one gains energy by breaking up the binding.

References

- [1] Finster, F., Smoller, J., and Yau, S.-T., gr-qc/9801079, *Phys. Rev. D* 59 (1999) 104020
- [2] Finster, F., hep-th/9703083, *J. Math. Phys.* 39 (1998) 6276-6290
- [3] Smoller, J., “Shock Waves and Reaction-Diffusion Equations,” 2nd ed., Springer (1994)
- [4] Bjorken, J. and Drell, S., “Relativistic Quantum Mechanics,” McGraw-Hill (1964)
- [5] Cooperstock, F. I., in “The Electron – New Theory and Experiment,” eds. D. Hestenes. and A. Weigartshofer, Kluwer Academic, Dordrecht (1991)
- [6] Bohun, C. S., MSc Thesis, University of Victoria (1991)
- [7] Lisi, G., “A Solution of the Maxwell-Dirac Equations in 3 + 1 Dimensions,” *J. Phys. A* 28 (1995) 5384

- [8] Adler, R., Bazin, M. and Schiffer, M., “Introduction to General Relativity,” McGraw-Hill (1975)

Mathematics Department,
Harvard University,
Cambridge, MA 02138 (FF & STY)

Mathematics Department,
The University of Michigan,
Ann Arbor, MI 48109 (JS)

email: felix@math.harvard.edu, smoller@umich.edu, yau@math.harvard.edu